Dual Spaces (Examples)

Example (A) :- Prove that a normed linear space is Separable if its Conjugate (or dual) space is Separable.

Solution :- Let N be a normed linear space whose conjugate space N^* is separable. Consider the set

$$S = \{ f : f \in N^*, \|f\| = 1 \}.$$

Since every subspace of a metric space is separable, S must be separable. Hence by definition of separability S contains countable dense subset, say $A = \{f_1, f_2, \longrightarrow\}$. Since each $f_n \in S$ we have $\|f_n\| = 1$ for all n.

Since $\|f_n\| = \sup\{|f_n(x)| : \|x\| = 1\}$, for each n, there must exist some vector x_n with $\|x_n\| = 1$ such that

$$|f_n(x_n)| > \frac{1}{2}$$

[If such x_n did not exist, this would contradict the fact that $\|f_n\| = 1$].

Let M be the closed linear subspace in N generated by the sequence $\langle x_n \rangle$, we assert that $M = N$. Suppose $M \neq N$ and let $x_0 \in N - M$

i.e. $x \notin M$. Then there exists an $F \in N^*$ such that

$$\|F\| = 1,\ F(x_0) \neq 0 \text{ and } F(x) = 0 \text{ if } x \in M.$$

Since $\|F\| = 1$, $F \in S$ and since each $x_n \in M$, we have

$F(x_n) = 0 \ (n = 1, 2, 3, \longrightarrow)$. Now.

$$\frac{1}{2} < |f_n(x_n)| = |(f_n(x_n) - F(x_n)) + F(x_n)|$$
$$\leq |f_n(x_n) - F(x_n)| + |F(x_n)|$$
$$= |(f_n - F)(x_n)| \quad [\because F(x_n) = 0]$$
$$\leq \|f_n - F\| \, \|x_n\| = \|f_n - F\| \{\because \|x_n\| = 1\}.$$

Thus $\|f_n - F\| \geq \frac{1}{2}$ for all n.

Now, since A is dense in S, every point of S is an adherent point of A so that each sphere centred at arbitrary $f \in S$ must contain a point of A. But the open sphere $\{f : \|f - F\| < \frac{1}{2}\}$ centred at $F \in S$ contains no point of A. We thus arrive at a contradiction and so we must have $M = N$. It then follows that the set of all linear combinations of the x_n's whose coefficients are rational or if N is complex have rational real and imaginary parts, constitute a countable set every where dense in N and consequently N is separable.

Example (8): — Give an example to show that the conjugate (dual) space of a separable space need not be separable.

Solution: — Consider the space l_1. We know that $l_1^* = l_\infty$. Our task is to show that l_1 is separable and l_∞ is not.

We first show that l_p is separable if $1 \leq p \leq \infty$. To see this we first introduce some terminology. A point $x \in l_p$ will be called rational if $x = \langle x_n \rangle$ and each x_n is rational. If the field is complex, x_n is called rational if its real and imaginary parts are rational. A rational point $x = \langle x_n \rangle$ will be called point of finite type if the set of n for which $x_n \neq 0$ is finite.

The set S of rational points of finite type is evidently countable. It is also everywhere dense in l_p.

ie. every point of l_p is an adherent point of S.

To show this for each $x = \langle x_n \rangle \in l_p$ and each $\epsilon > 0$ we must produce an element $y \in S$ such that

$$\|y - x\| < \epsilon$$

So let $x = \langle x_n \rangle \in l_p$ and let $\epsilon > 0$ be given by definition of l_p $\sum_{n=1}^{\infty} |x_n|^p$ is convergent and so there exists a positive integer m

$$\text{Such that } \sum_{n=m+1}^{\infty} |x_n|^p > \epsilon^p/2 \quad \text{——— (1)}$$

We now choose a rational point of finite type (i.e. a point of S) say $y = \langle x_n \rangle$ as follows.

$$\text{if } n > m, \text{ let } r_n = 0 \quad \text{——— (2)}$$

$$\text{if } n = 1, 2, \dots, \text{ let } \|r_n - x_n\|^p < \frac{\epsilon^p}{2m} \quad \text{——— (3)}$$

Then $\|y - x\|^p = \sum_{n=1}^{\infty} |r_n - x_n|^p = \sum_{n=1}^{m} |r_n - x_n|^p + \underbrace{\sum_{n=m+1}^{\infty} |x_n|^p}_{\text{by (2)}}$

$$< m \frac{\epsilon^p}{2m} + \frac{\epsilon^p}{2} = \epsilon^p \text{ by (1) and (3)}$$

So that $\|y - x\| < \epsilon$. Then l_p is separable when $1 \le p < \infty$.

In particular l_1 is separable.

We next show that l_∞ is not separable. let $\langle x_n \rangle$ be any countable set in l_∞ with

$$x_n = \langle x_1^m, x_2^m, \text{——} \rangle$$

And let $x = \langle z_1, z_2, \ldots z_k \ldots \rangle$ be an element of l_p defined by $z_k = x_k^{(k)} + 1$ if $|x_k^{(k)}| \leq 1$

and $z^k = 0$ if $|x_k^{(k)}| > 1$.

Then the kth component of $z - x_k$ is $z_k - x_k^{(k)}$ and $|z_k - x_k^{(k)}| \geq 1$ so that

$\|z - x_k\| \geq 1$.

Hence z cannot be an adherent point of the set $\{x_n\}$ and consequently $\{x_n\}$ cannot be dense in l_∞. It follows that l_∞ is not separable.

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